Name:________________________________________________Per:____________

AP Physics C

Semester 1 - Mechanics

# Unit 6
# Rotational Motion
# Workbook

# Unit 6 Rotational Motion

Supplements to Text Readings from
Fundamentals of Physics by Halliday, Resnick, & Walker
**<u>Chapter 11, 12 & 13</u>**

# Unit 6 – Objectives and Assignments

Text: Fundamentals of Physics by Halliday, Resnick, & Walker **Chapter 11, 12 & 13**

## I. Torque and Rotational Statics

a. Students should understand the concept of torque so they can:
   (1) Calculate the magnitude and sense of the torque associated with a given force.
   (2) Calculate the torque on a rigid body due to gravity.

b. Students should be able to analyze problems in statics so they can:
   (1) State the conditions for translational and rotational equilibrium of a rigid body.
   (2) Apply these conditions in analyzing the equilibrium of a rigid body under the combined influence of a number of coplanar forces applied at different locations.

## II. Rotational Kinematics

a. Students should understand the analogy between translational and rotational kinematics so they can write and apply relations among the angular acceleration, angular velocity, and angular displacement of a body that rotates about a fixed axis with constant angular acceleration.

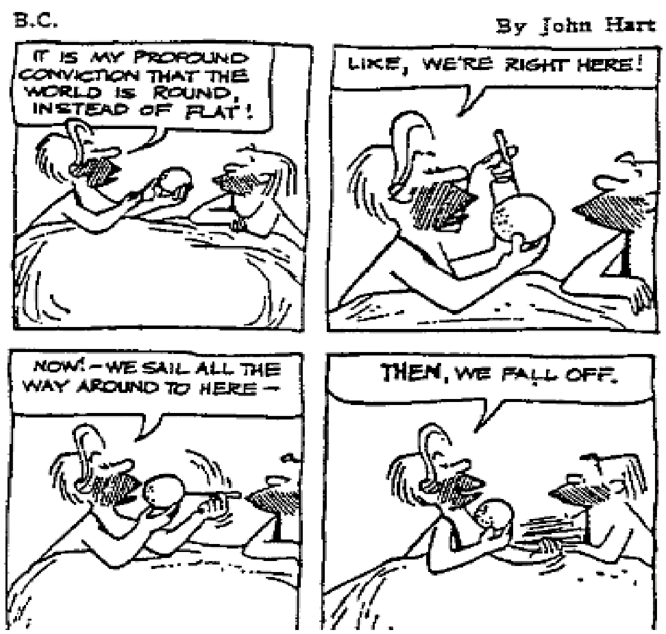
b. Students should be able to use the right-hand rule to associate an angular velocity vector with a rotating body.

## III. Rotational Inertia

a. Students should develop a qualitative understanding of rotational inertia so they can:
   (1) Determine by inspection which of a set of symmetric bodies of equal mass has the greatest rotational inertia.
   (2) Determine by what factor a body's rotational inertia changes if all its dimensions are increased by the same factor.

b. Students should develop skill in computing rotational inertia so they can find the rotational inertia of:
   (1) A collection of point masses lying in a plane about an axis perpendicular to the plane.
   (2) A thin rod of uniform density, about an arbitrary axis perpendicular to the rod.
   (3) A thin cylindrical shell about its axis, or a body that may be viewed as being made up of coaxial shells.
   (4) A solid sphere of uniform density about an axis through its center.

c. Students should be able to state and apply the parallel-axis theorem.



## IV. Rotational Dynamics

a. Students should understand the dynamics of fixed-axis rotation so they can:
   (1) Describe in detail the analogy between fixed-axis rotation and straight-line translation.
   (2) Determine the angular acceleration with which a rigid body is accelerated about a fixed axis when subjected to a specified external torque or force.
   (3) Apply conservation of energy to problems of fixed-axis rotation.
   (4) Analyze problems involving strings and massive pulleys.

b. Students should understand the motion of a rigid body along a surface so they can:
   (1) Write down, justify, and apply the relation between linear and angular velocity, or between linear and angular acceleration, for a body of circular cross-section that rolls without slipping along a fixed plane, and determine the velocity and acceleration of an arbitrary point on such a body.
   (2) Apply the equations of translational and rotational motion simultaneously in analyzing rolling with slipping.
   (3) Calculate the total kinetic energy of a body that is undergoing both translational and rotational motion, and apply energy conservation in analyzing such motion.

## V. Angular Momentum and Its Conservation

a. Students should be able to use the vector product and the right-hand rule so they can:
   (1) Calculate the torque of a specified force about an arbitrary origin.
   (2) Calculate the angular momentum vector for a moving particle.
   (3) Calculate the angular momentum vector for a rotating rigid body in simple cases where this vector lies parallel to the angular velocity vector.

b. Students should understand angular momentum conservation so they can:
   (1) Recognize the conditions under which the law of conservation is applicable and relate this law to one- and two-particle systems such as satellite orbits or the Bohr atom.
   (2) State the relation between net external torque and angular momentum, and identify situations in which angular momentum is conserved.
   (3) Analyze problems in which the moment of inertia of a body is changed as it rotates freely about a fixed axis.
   (4) Analyze a collision between a moving particles and a rigid body that can rotate about a fixed axis or about its center of mass.

**Mechanics Unit 6 Homework**

| | |
|---|---|
| Chapter 11 | #7, 9, 19, 22, 23, 35, 37, 39, 41, 45, 46, 50, 56, 57, 62, 63, 67, 71, 72, 79, 81, 82 |
| Chapter 12 | #14, 15, 17, 23, 24, 31, 32, 34, 39, 41, 45, 47, 55, 61, 63, 65, 67 |
| Chapter 13 | #23, 26, 33, 36 |

Don't bother reading Ch. 13.5 & 13.6

# Translating Linear Equations to Rotational Equations

| Linear Motion | Rotational Motion |
|---|---|
| Distance x = | □ = |
| Displacement $\Delta x$ | $\Delta$□ = |
| Ave. Velocity $v_{ave}$ = | □$_{ave}$= |
| Instant. Velocity $v_{inst}$ = | □$_{inst}$ = How are v and □ related? |
| Ave. Acceleration $a_{ave}$ = | □$_{ave}$ = |
| Instant. Acceleration $a_{inst}$ = | □$_{inst}$ = How are a and □ related tangentially? |
| | How are a and □ related radially? |
| | Right Hand Rule |

| IRRELEVANT Linear Kinematics Equations | IRRELEVANT Rotational Kinematics Equations |
|---|---|
| x | □ |
| v | □ |
| t | t |
| a | □ |
| $v_o$ | □$_o$ |

| | |
|---|---|
| $KE_t$ = | $KE_r$ = |
| Net Force □**F** = | Net Torque □□= |
| Work- Energy | Work- Energy |
| Work = ∫ | Work = ∫ |
| Power = | Power = |
| Linear Momentum **p** = | Angular Momentum **L** = |
| Real 2nd Law □**F** = | 2nd Law □□= |

AP Physics C

Unit 6 - Rotational Motion

# Rotational Kinematics

## 1. Rotational Kinematics with Rotating Disk

A circular disk (like a CD, wheel, or galaxy disk), starting from rest, rotates with an angular acceleration given by

$$\alpha = (3 + 4t) \quad \text{rad/s}^2$$

a) Derive the expression for the angular speed[1] as a function of time.

b) Derive the expression for the angle[2] the wheel turns through a function of time.

c) Determine[3] $\alpha$, $\omega$ and $\theta$ at $t = 2$ s.

d) If the disk has a radius of 3 m, determine the linear speed[4] and the radial[5] and tangential[6] components of the linear acceleration of a point on the rim of the disk at $t=2$ s.

---

[1] $3t + 2t^2$
[2] $1.5t^2 + 0.67t^3$
[3] 11 rad/s$^2$, 14 rad/s, 34/3 rad
[4] 42 m/s
[5] 588 m/s$^2$
[6] 33 m/s$^2$

## 2. Merry-Go-Round Dynamics

A kid is standing on a Merry-Go-Round 5 meters from its axis of rotation. Starting from rest, the M-G-R accelerates uniformly. After 8 seconds, its angular speed is 0.08 rev/sec. (Hint: Change rev/sec to rads/sec first.)

a) At 8 seconds, find the angular speed[7].

b) At 8 seconds, find the linear speed[8].

c) At 8 seconds, find the angular acceleration[9].

d) At 8 seconds, find the centripetal acceleration[10].

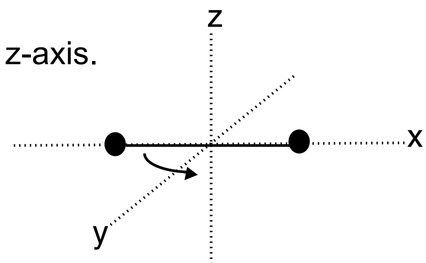
e) At 8 seconds, find the tangential acceleration[11].

[7] 0.5 rad/s
[8] 2.5 m/s
[9] 0.062 rad/$s^2$
[10] 1.25 m/$s^2$
[11] 0.32 m/$s^2$

## 3. Atomic Rotational Motion

Consider the diatomic molecule oxygen, $O_2$, which is rotating in the xy plane about the z-axis passing through its center, perpendicular to its length. The mass of each oxygen atom is $2.66 \times 10^{-26}$ kg, and at room temperature, the average separation between the two oxygen atoms is $d = 1.21 \times 10^{-10}$ m (the atoms are treated as point masses).

a) Calculate the moment of inertia[12] of the molecule about the z-axis.

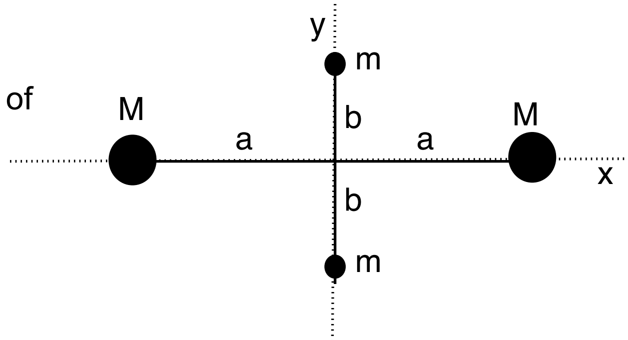



b) If the angular speed of the molecule about the z axis is $4.6 \times 10^{12}$ rad/s, what is the rotational Kinetic Energy[13]?

[12] $1.95 \times 10^{-46}$ kgm$^2$
[13] $2.06 \times 10^{-21}$ J

## 4. Rotating Point Masses

Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane as shown below.




a) If the rotation of the system occurs about the y-axis with an angular speed □, find the moment of inertia[14] about the y-axis.

b) Calculate the rotational kinetic energy[15] about the y-axis.

c) Suppose the system rotates in the xy plane about an axis through the z-axis. Calculate the moment of inertia[16] about the z-axis.

d) Calculate the rotational KE[17] about the z-axis.

e) What conclusions can you make when comparing the answers for a and c?

f) What conclusions can you make when comparing the answers for b and d?

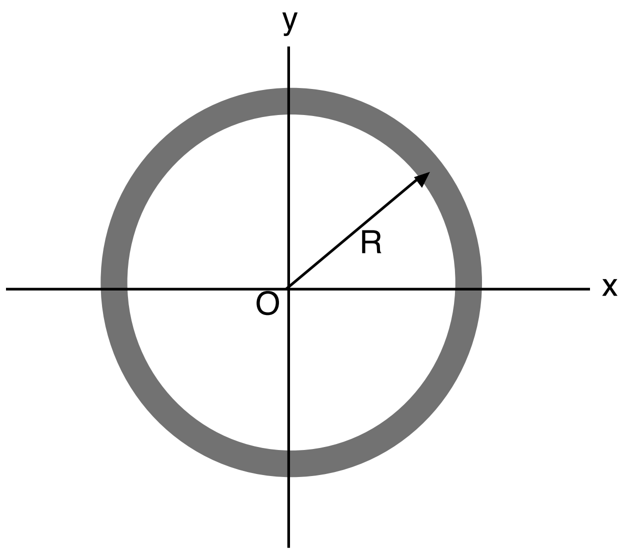
---

[14] $2Ma^2$
[15] $Ma^2$□$^2$
[16] $2Ma^2 + 2mb^2$
[17] $(Ma^2 + mb^2)$ □$^2$

# Moment of Inertia of Non-Particles

## 1. Spinning Uniform Hoop

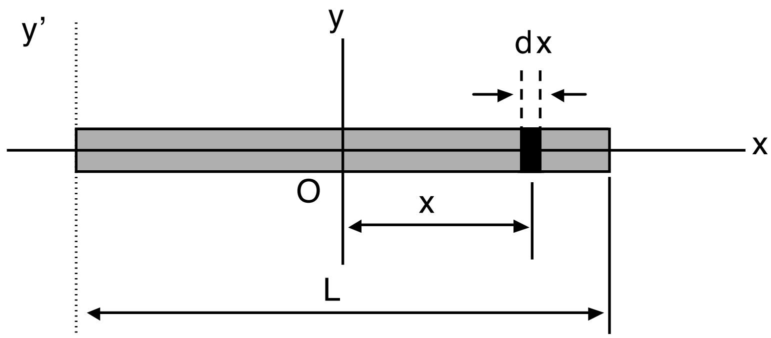
Find the moment of inertia[18] of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop, through its center as shown below.




[18] $MR^2$

## 2. Spinning Uniform Rod (Ok I don't care if you like these or not)

a) Calculate the moment of inertia[19] of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod (the y-axis) and passing through its center of mass.

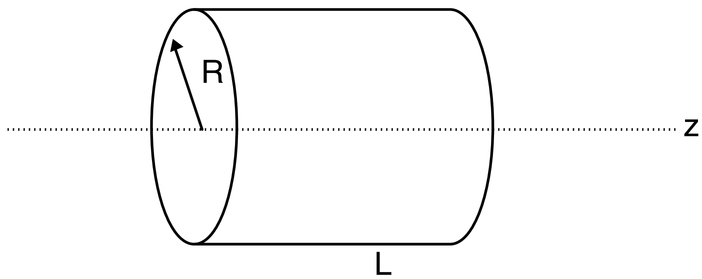



b) Calculate the moment of inertia[20] of the above uniform rigid rod through one end (the y'-axis).

[19] $ML^2 / 12$
[20] $ML^2/3$

## 3. Spinning Uniform Solid Cylinder

A uniform solid cylinder has a radius R, mass M, and length L. Calculate the moment of inertia[21] about its central axis (the z-axis).

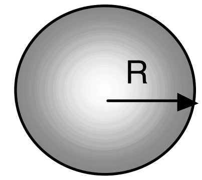



[21] $MR^2 / 2$

## 4. Solid Sphere

This one is HARD and a challenge for those who seek mathematical adventure!
Show that the moment of inertia for a solid sphere of mass M and radius R is given by

$$I_{\text{solid sphere}} = (2/5)\, MR^2$$




Hints: 1) Cut the sphere into 2 hemispheres.
2) Divide the hemisphere into thin disks with thickness dx.
3) Change dm to a function of dV for thin disks.
4) Integrate for half of the sphere, then multiply answer by 2 for whole sphere.

Note that if you divide the sphere into thin hollow spheres and add up all the thin spheres, you will get $I = (\mathbf{3}/5)\, MR^2$ which is incorrect.

# Parallel-Axis Theorem (Know This!)

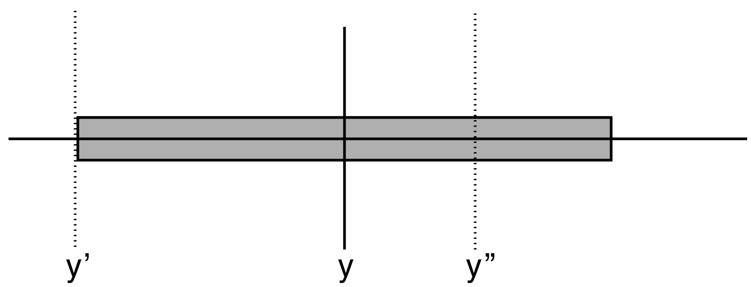
Purpose - How do you find the moment of inertia about an arbitrary axis, not just through the axis of the center of mass?

Solution - Proof on Pg. 250 of book. READ IT!

| | |
|---|---|
| Parallel-Axis Theorem<br><br>$I = I_{cm} + MD^2$ | $I$ is moment of inertia about the new arbitrary axis<br>-this new axis MUST be PARALLEL to $I_{cm}$<br>$I_{cm}$ is the moment of inertia about the center of mass<br>-most are listed on Pg. 249 of text<br>M = total mass of object<br>D = distance from center of mass axis to new arbitrary axis |

## 1. Parallel Axis Theorem and Spinning Uniform Rod

a) Given that the moment of inertia of a thin uniform rod about the center of mass ( y )is $I_{cm} = ML^2 / 12$, use the parallel axis theorem and determine the moment of inertia[22] about one of the ends (y').

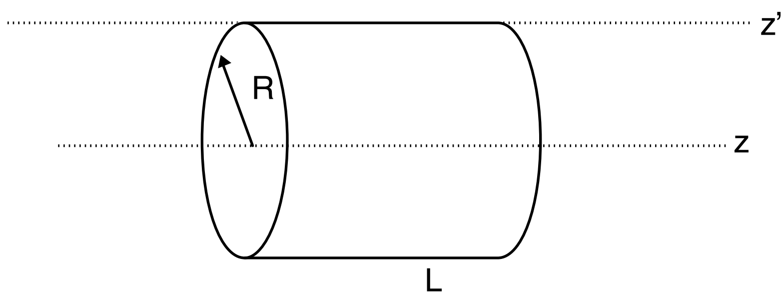



b) Use the parallel axis theorem and determine the moment of inertia[23] about y" = L/4.

[22] $ML^2 / 3$
[23] $7ML^2 / 48$

## 2. Parallel Axis Theorem and Spinning Solid Cylinder

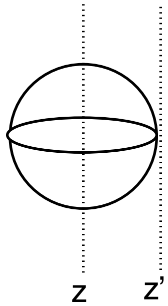
Use the parallel axis theorem and determine the moment of inertia[24] of a solid cylinder about an axis tangent to the outer edge (z').




[24] $3MR^2 / 2$

## 3. Parallel Axis Theorem & Spinning Solid Sphere

Use the parallel axis theorem and determine the moment of inertia[25] of a solid sphere about an axis tangent to the outer edge (z').




[25] $7MR^2 / 5$

# Some Stuff about Rolling Things

In the past, we have considered the principle of conservation of mechanical energy for objects that possessed only translational kinetic energy (i.e. the object did NOT spin or rotate). However, a body that is rotating also possesses ROTATIONAL kinetic energy. The rotational kinetic energy $KE_r$ of an object having a moment of inertia and an angular speed □ is given by

$KE_r$ = ______________________

Thus we can say that an object rolling across the floor possesses total kinetic energy given by the sum of its rotational and translational kinetic energies.

If an object has a mass m, moment of inertia I, translational speed v, and angular speed □, then its $KE_{total}$ is given by

$KE_{total}$ = _______________ + _______________

For an object rolling down an incline, one way of stating the principle of conservation of energy is that the total mechanical energy of the system when the object is released equals the total mechanical energy of the system when the object reaches the bottom of the incline. Suppose the object starting from rest, rolls down the incline so that its center of mass is lowered a vertical distance h. In terms of m, h, I, v, and □, then conservation of energy can be expressed as

________________ = ___________________ + ____________________ (Eqn. A)

**Disks** - If the object is a disk of radius R and mass m, it moment of inertia is given by

$I_{disk}$ = _______________________ and therefore, $KE_r$ = ________________________

Equation A for the case of a rolling disk becomes

________________ = ___________________ + ____________________ (Eqn. B)

The angular speed □ can be expressed in terms of the speed of a point on the rim of the disk $v_{rim}$ and R by

□ = ____________ (Eqn. C)

As the disk rolls through one revolution, the point on the rim and the center of mass of the disk move a distance 2πR along the incline during the same time interval. Hence, $v_{rim} = v_{translational} = v$ and equation B can be written in terms of m, v, g, and h as

________________ = ___________________ + ____________________ (Eqn. D)

Solving equation D for v will give the translational speed of the disk after it falls through a vertical distance h starting from rest. Solving for velocity in Eqn. D gives

v = _________________________

Note that the speed does NOT depend on (mass, size of disk, gravity, angular speed). ALL disks regardless of their mass or radius, if released from the same place at the same time will roll down the incline side-by-side! Check it out for yourself!

**Hoop & Sphere -** Begin by writing the statement of conservation of energy, follow the same steps used for the disk and derive the expression for the translational speed of a hoop and a sphere rolling down an incline where its center of mass is lowered a vertical height h.

**HOOP:** Statement of Conservation of Energy for Hoop

_________________ = ____________________ + _____________________

$v_{hoop}$ = _________________

**Sphere:** Statement of Conservation of Energy for Sphere

_________________ = ____________________ + _____________________

$v_{sphere}$ = _________________

Does the translational speed of the hoop or sphere at the bottom of the incline depend on its mass of size? _____________

Suppose a disk, hoop, and sphere are released at the same time from the same level along an incline. In what order will they reach the bottom of the incline? Why?

1st = ____________________, 2nd = ______________________, 3rd = ___________________

If the size and/or masses are changed, will the order of arrival at the bottom change? ________

AP Physics C

Unit 6 - Rotational Motion

# Cross Products

(useful for torque and angular momentum)

Read pg. 46 & 47 of your textbook for the details of the following:

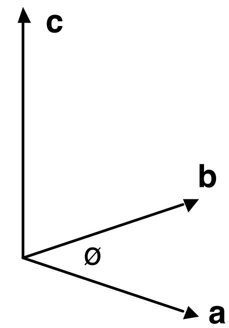
The cross product **a x b** creates a 3rd vector **c** whose magnitude is

$c = ab \sin\phi$ $\qquad$ $c = ab \sin\phi$

where ø is the angle between **a** and **b.**

$\mathbf{a \times b} = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$

$$\hat{k} = \hat{i} x \hat{j} = -(\hat{j} x \hat{i})$$
$$\hat{j} = \hat{k} x \hat{i} = -(\hat{i} x \hat{k})$$
$$\hat{i} = \hat{j} x \hat{k} = -(\hat{k} x \hat{j})$$

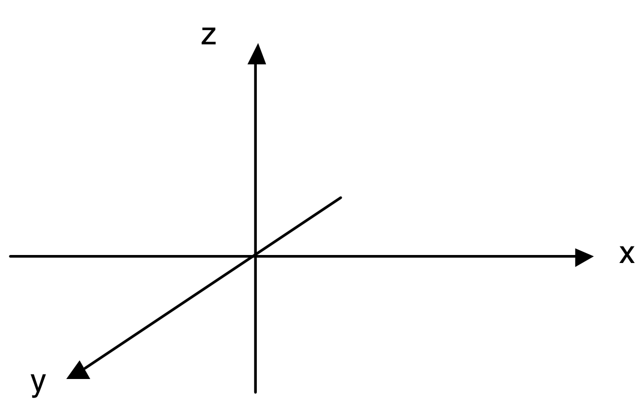



Direction of **c** is given by **right hand rule**.

**a x b = - (b x a)**

1. Two vectors are given by **a** = $3\hat{i} + 5\hat{j}$ and **b** = $2\hat{i} + 4\hat{j}$.
   Find the cross product **a x b.**

Can you draw the vectors?



2. Two vectors **a** and **b** have components, in arbitrary units

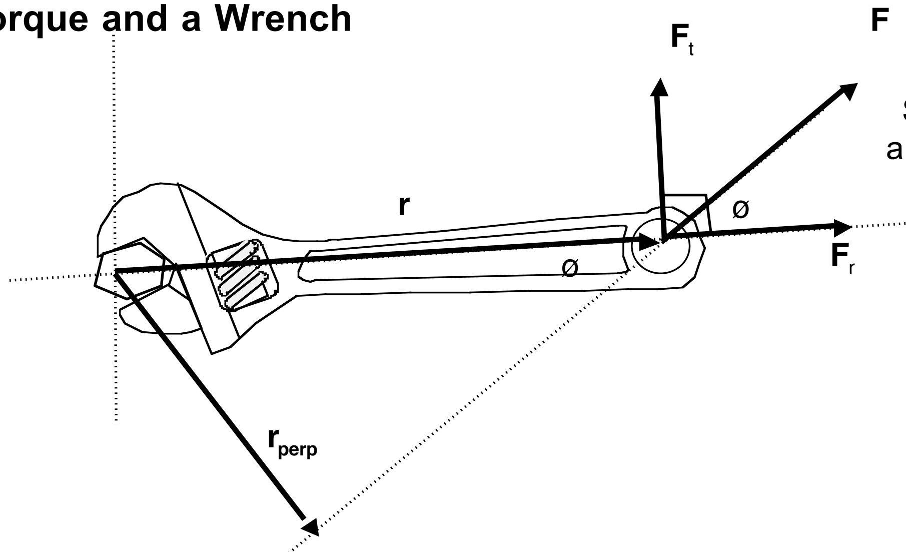
   i) Find the angle[26] between **a** and **b.**

   ii) Find the component of a vector **c** that is perpendicular to **a**, is in the xy plane, and has a magnitude of 5 units.

[26] 57°

# Torque □

## 1. Torque and a Wrench




Suppose you apply a force F on a wrench to turn a bolt as shown on the left.

F can be broken up into its perpendicular parts $F_t$ and $F_r$.

Torque for rotational motion is the analogous counterpart of force in linear motion.

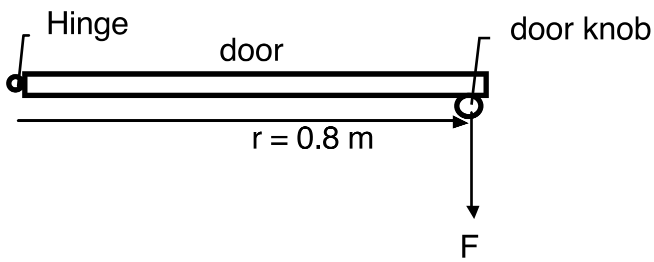
What does torque depend on? In other words, what do you have to do to turn the bolt with minimum effort?

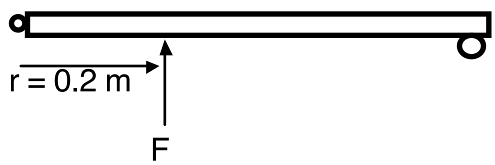
## 2. Torque and Doors

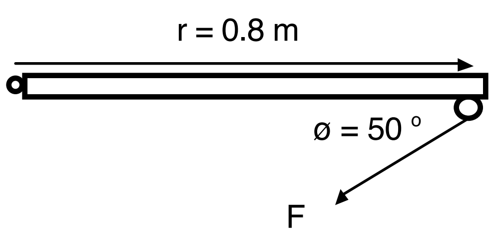
Suppose you open a door with a force of 55 N with different lever arms.

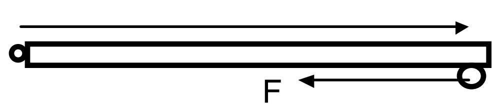
Bird's eye view of a door

Calculate the magnitude and indicate the direction of the torque with each situation.



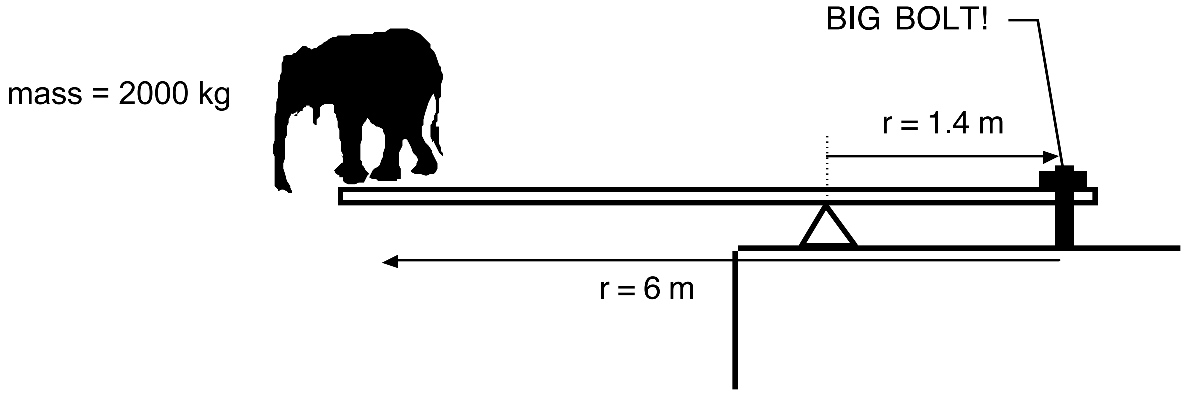
# Rotational Equilibrium

$\Sigma F = 0 \quad \Sigma \tau = 0$

at **equilibrium (Ch. 13)**
with translational and rotational forces

## 1. A Diving Elephant

Below is a diving board with an elephant just ready to dive.




Determine the force of the fulcrum[27] and the force of the bolt[28] on the diving board.

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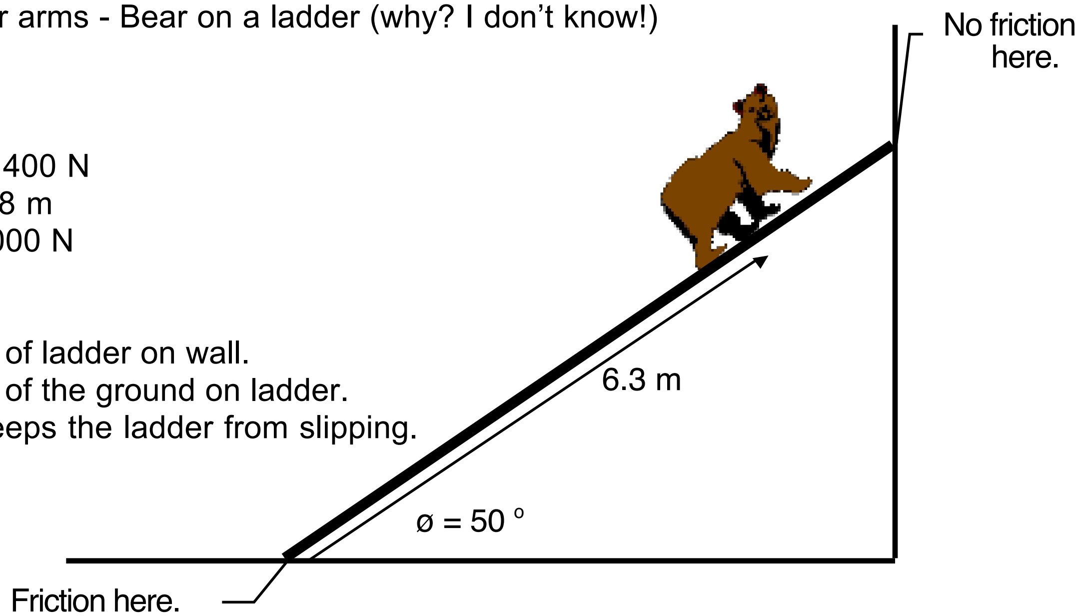
[27] 84000 N
[28] 64400 N

## 2. A Climbing Bear

Calculating lever arms - Bear on a ladder (why? I don't know!)

Ladder weight = 400 N
Ladder length = 8 m
Bear weight = 2000 N

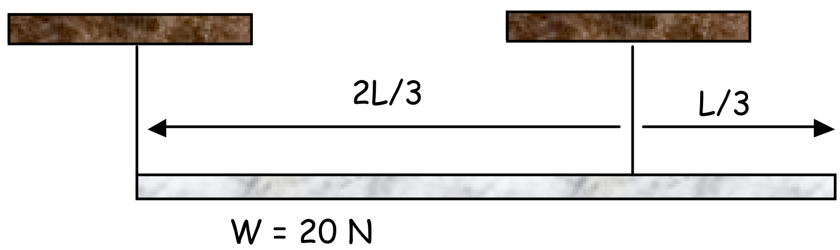
a) Find the force of ladder on wall.
b) Find the force of the ground on ladder.
c) Find $\mu_s$ that keeps the ladder from slipping.



## 3. Hanging rod, hinges, and masses

What is/are the tension(s) in the each of the situations below?

a)



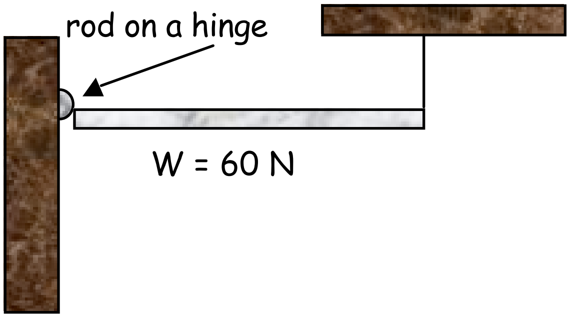

Hanging rod on two strings.

b)


Hanging rod on two strings with added mass.

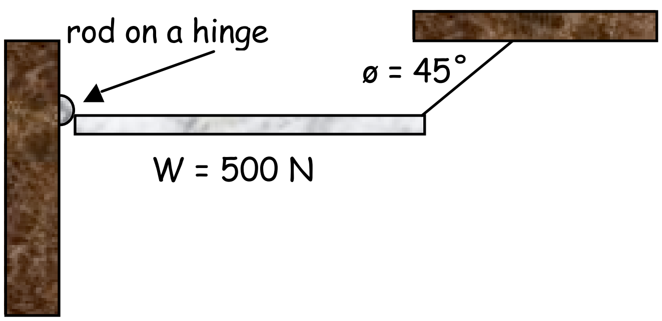
W = 160 N

c)

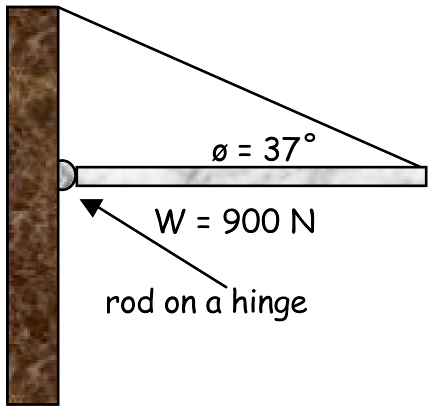



Hint for hinges: Hinges apply a reaction force on an attached object. Think about how the object acts on the hinge and the hinge will act equally and opposite to that force.

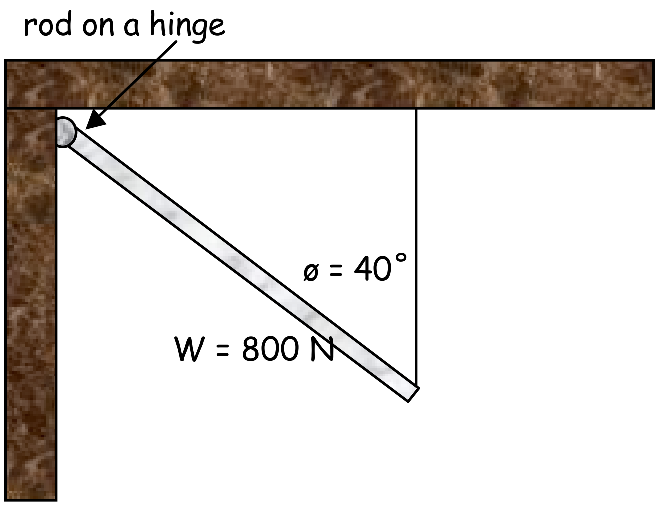
d)



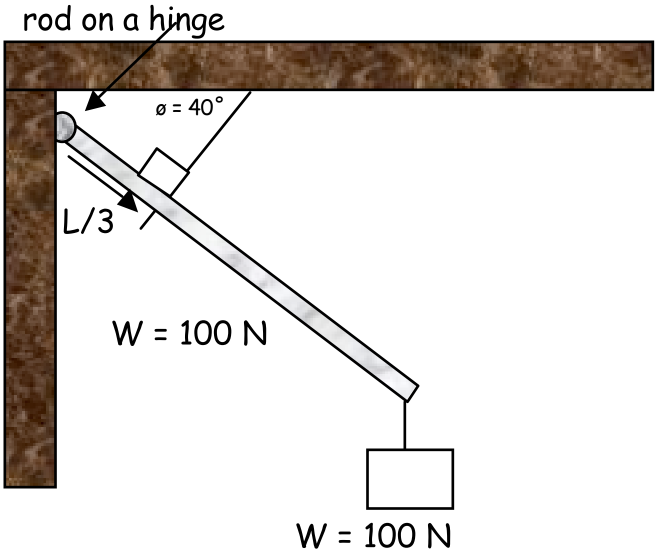
e)




f)



g)

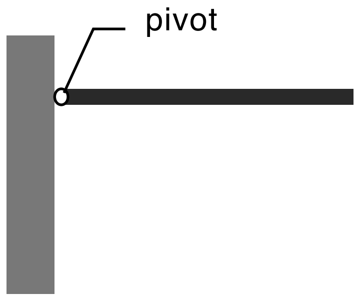


# Rotational Non-Equilibirum

$$\boxed{\Sigma\tau = I\alpha}$$

## 1. Rotational Free Fall

A uniform rod of length L and mass M is free to rotate about a frictionless pivot at one end in a vertical plane as shown below. The rod is released from rest in the horizontal position.




a) If you were to determine the angular acceleration of the rod and initial linear acceleration of the right end of the rod, why could you NOT use the rotational kinematics equations?

b) What is the initial angular acceleration[29] of the rod and initial linear acceleration[30] of the right end of the rod?

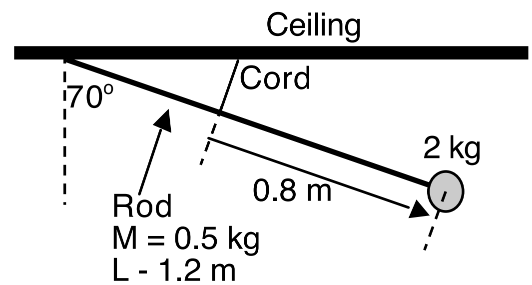
c) What would happen if you placed a coin on the right end of the rod and let the rod drop?

d) At what point on the rod is the initial linear acceleration the same as the acceleration due to gravity?

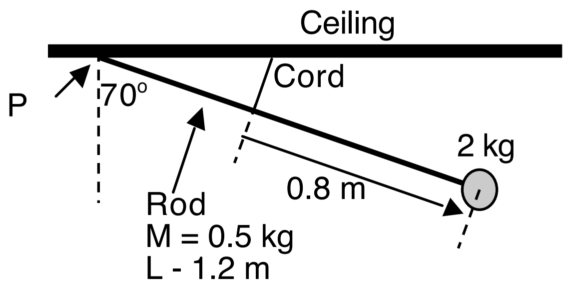
[29] $(3g)/(2L)$
[30] $3g/L$

## 2. Equilibrium/Rotation

In the situation shown at the right, a rod with a mass attached to its lower end is in equilibrium with its upper and resting unattached to the ceiling. The cord is perpendicular to the rod.




a) Determine the minimum value of the coefficient of static friction $\mu_s$ between the rod and the ceiling that will prevent the rod from slipping.

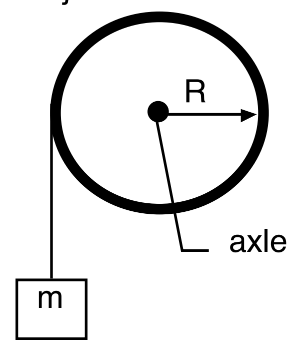



b) Instead of the rod resting on the ceiling, let it be attached at point P to the ceiling with a frictionless hinge. Suppose the cord is cut.

i) Determine the moment of inertia of the rod-mass system as it rotates about the hinge.
ii) Determine the net torque on the system.
iii) Determine the angular acceleration □ of the system.
iv) Determine the angular speed □ as the system swings through the vertical.

## 3. Dynamics of Pulley and Hanging Mass

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle as shown on the right. A light cord wrapped around the wheel supports an object of mass m.




a) Calculate the linear acceleration[31] of the object, angular acceleration[32] of the wheel and the tension[33] in the cord.

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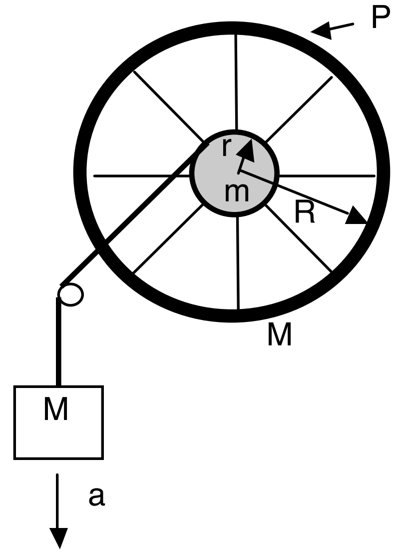
[31] $g/(1 + I/mR^2)$
[32] $g/(R + I/mR)$
[33] $mg/(1 + mR^2/I)$

## 4. Another Wheel-Disk and Hanging Mass System




Given: $m = 0.2$ kg  $r = 0.1$ m
$M = 1.5$ kg  $R = 0.3$ m

Neglect the following:
- -Friction in the axle & pulley
- -Mass of the spokes and pulley

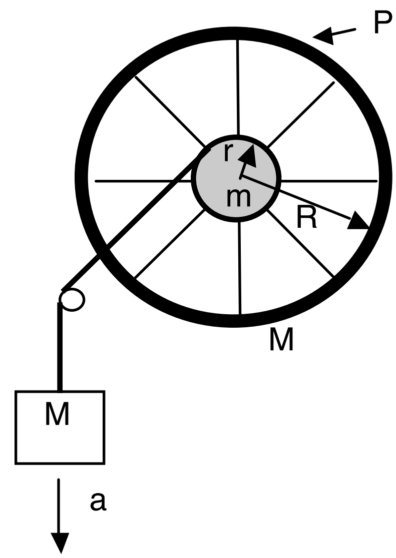
Determine each of the following:

a) The moment of inertia of the wheel-disk system
b) Angular acceleration of the disk and linear acceleration of M
c) Magnitude and direction of the friction force $f_p$ applied at point P if the system is to be in equilibrium

d) Angular acceleration of the disk and linear acceleration of M if the friction force $|\square_p|$ applied at point P is 1.0 N

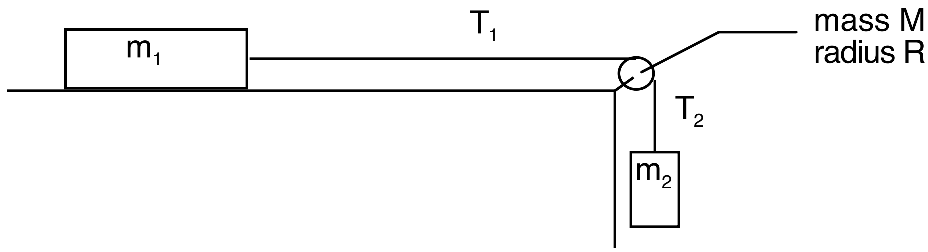
e) Tension T in the cord if the friction force $|\square_p|$ applied at point P is 1.0 N

f) Time required for the block to fall 2 meters after being released if the friction force $|\square_p|$ applied at point P is 1.0 N



## 5. The REAL 2 Mass and Pulley System

Consider the system below. Because friction exists between the string and the pulley, a torque is placed on the pulley unlike the perfect situation where the pulley simply redirects the tension force. This means we cannot assume the tension in the string is constant throughout. Find the acceleration[34] and both tensions[35] in the string.

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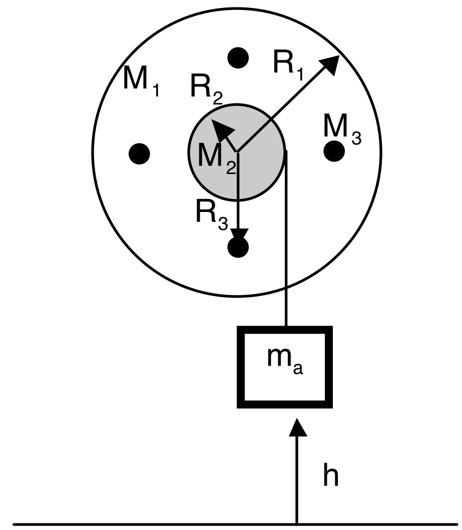
[34] $mg/(m_2 + m_1 + M)$

[35] $T_1 = m_1m_2g/(m_2 + m_1 + M)$ $T_2 = (m_1 + M)m_2g/(m_2 + m_1 + M)$

# Rotation Review




A disk of radius $R_1$ and mass $M_1$ is free to rotate without friction about a horizontal axle through its center, perpendicular to its face. Attached to it is a smaller disk of radius $R_2$ and mass $M_2$ as shown on the right. Also attached to the disk at a distance $R_3$ from the axle are four small pucks each of mass $M_3$. A "massless" string is wrapped around the smaller disk and a mass m is attached to its hanging end. The mass $m_a$ is released from rest and falls through a vertical distance h.

| | | |
|---|---|---|
| Given: $R_1$ = 0.2 m | h = 0.5 m | $M_1$ = 0.25 kg |
| $R_2$ = 0.05 m | $m_a$ = 0.25 kg | $M_2$ = 0.10 kg |
| $R_3$ = 0.15 m | | $M_3$ = 0.25 kg |

a) Calculate the moment of inertia[36] of the system about its axis of rotation.

b) Find the angular speed[37] of the system after the mass $m_a$ has fallen through a vertical distance h.

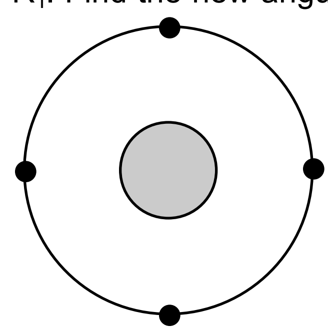
c) Find the angular acceleration[38] of the system. (Think kinematics!)

[36] 0.028 kg•m$^2$
[37] 9.35 rad/s
[38] 4.37 rad/s$^2$

d) Using FBDs and $\Sigma\tau = I\alpha$, find the tension[39] in the string as $m_a$ falls.

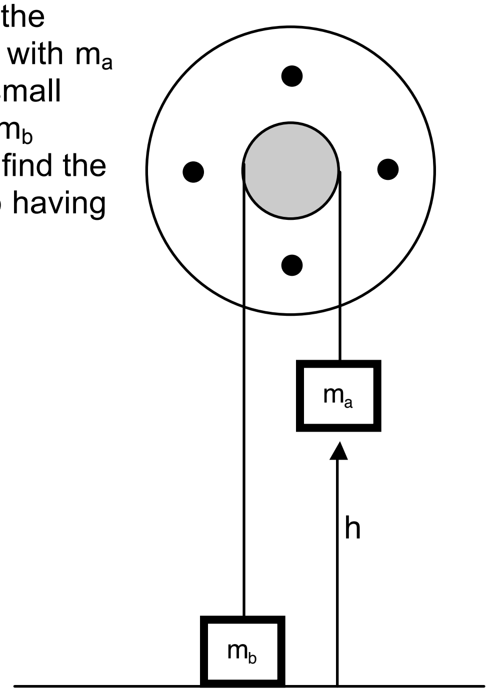
e) Using FBDs and $\Sigma F = ma$, again find the tension in the string as $m_a$ falls.

f) At the instant m has fallen vertical distance h, the string becomes detached from the rotating system. Shortly after the string comes loose, the four pucks suddenly slide radially outward and come to rest with their centers at the edge of the large disk so that $R_3 = R_1$. Find the new angular speed[40] of the rotating system.



[39] 2.44 N
[40] 5.82 rad/s

g) Now the pucks are back to their original starting position and the whole system is at rest. Suppose that in addition to the string with $m_a$ attached, another "massless" string is wrapped around the small disk in the opposite direction and that the string has a mass $m_b$ attached to its hanging end as shown on the right. If $m_a > m_b$, find the value of the second mass[41] $m_b$ that will result in the system to having an angular acceleration of 3 rad/s$^2$.




h) If $m_b$ is at rest on the surface as shown in the diagram above when $m_a$ is released, use the principle of conservation of energy to find the speed[42] of $m_a$ just before it strikes the surface.

[41] 0.077 kg
[42] 0.39 m/s